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The Rikitake Two-Disk Dynamo System and Domains with Periodic Orbits

Yorick Hardy[1] **and W.-H. Steeb**[1]

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The Rikitake two-disk dynamo system is a simple model to describe the earth's magnetic field. We derive the conditions to find periodic orbits of this system using an ellipsoid bounding condition. We prove that the conditions cannot be satisfied.

The Rikitake two-disk dynamo model consists of two connected identical frictionless disk dynamos (Cook and Roberts, 1970). The dynamos are driven by identical torques G to maintain their motions in the face of Ohmic losses in the coils and disks. The equations describing the system are given by the nonlinear dynamical system

$$L\frac{dI_1}{dt} + RI_1 = M\Omega_1 I_2 \tag{1a}$$

$$L\frac{dI_2}{dt} + RI_2 = M\Omega_2 I_1 \tag{1b}$$

$$C\frac{d\Omega_1}{dt} = G - MI_1 I_2 \tag{1c}$$

$$C\frac{d\Omega_2}{dt} = G - MI_1 I_2 \tag{1d}$$

where I_1 and I_2 are the currents, Ω_1 and Ω_2 are the angular velocities, L is the self-inductance and R is the resistance associated with each dynamo and its connecting circuitry, M is the mutual inductance between the dynamo

[1] Rand Afrikaans University, P.O. Box 524, Auckland Park 2006, South Africa; e-mail: WHS@NA.RAU.AC.ZA.

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circuits, and C is the moment of inertia of a dynamo about its axis. System (1) can be cast into the form [1]

$$\frac{dX_1}{dt} = -\mu X_1 + Y_1 X_2 \tag{2a}$$

$$\frac{dX_2}{dt} = -\mu X_2 + (Y_1 - A) X_1 \tag{2b}$$

$$\frac{dY_1}{dt} = 1 - X_1 X_2 \tag{2c}$$

where

$$I_i := \left(\frac{G}{M}\right)^{1/2} X_i, \qquad \Omega_i := \left(\frac{GL}{CM}\right)^{1/2} Y_i, \qquad i = 1, 2$$

and

$$I_1 = \pm k\left(\frac{G}{M}\right)^{1/2}, \qquad I_2 = \pm k^{-1}\left(\frac{G}{M}\right)^{1/2}$$

$$\Omega_1 = k^2\left(\frac{R}{M}\right), \qquad \Omega_2 = k^{-2}\left(\frac{R}{M}\right)$$

$$t \to \frac{t}{\sqrt{\tau_e \tau_m}}, \qquad A = \mu(k^2 - k^{-2}), \qquad \mu > 0$$

$A > 0$ can be assumed; otherwise the roles of the dynamos are reversed. For $\mu = 0$ and $A = 0$ the system can be solved exactly. For $A = 0$ and μ arbitrary we find the first integral

$$I(X_1, X_2, Y, t) = (X_1^2 - X_2^2)e^{\mu t}$$

Next we derive the conditions for an ellipsoid bounding of the Rikitake two-disk dynamo system (2). We apply the following theorem (Krishchenko, 1997) to find the bounding conditions of a system of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^n$$

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))^T \in C^\infty(\mathbf{R}^n)$$

Theorem. All cycles of the system are contained in

$$\Omega_\psi = \{\mathbf{x} \mid \psi_{\inf} \le \psi \le \psi_{\sup}\}$$

where

$$\psi_{\inf} := \inf\{\psi(\mathbf{x}) \big| L_V\psi(\mathbf{x}) = 0\}$$
$$\psi_{\sup} := \sup\{\psi(\mathbf{x}) \big| L_V\psi(\mathbf{x}) = 0\}, \qquad \psi(\mathbf{x}) \in C^{\infty}(\mathbf{R}^n)$$

The vector field

$$V = \sum_{i=1}^{n} f_i(\mathbf{x}) \frac{\partial}{\partial x_i}$$

is associated with the autonomous dynamical system $d\mathbf{x}/dt = \mathbf{f}(\mathbf{x})$, and

$$L_V\psi(\mathbf{x}) := \sum_{i=1}^{n} f_i(\mathbf{x}) \frac{\partial \psi}{\partial x_i}$$

denotes the Lie derivative (Steeb, 1996a, b) with respect to V. For the proof we refer to Krishchenko (1997).

Now we apply this theorem to the Rikitake two-disk dynamo model. For the Rikitake two-disk dynamo system the corresponding vector field is

$$V = (-\mu X_1 + Y_1 X_2) \frac{\partial}{\partial X_1} + (-\mu X_2 + (Y_1 - A)X_1) \frac{\partial}{\partial X_2} + (1 - X_1 X_2) \frac{\partial}{\partial Y_1}$$

We assume that ψ is given by

$$\psi(X_1, X_2, Y_1) = \alpha X_1^2 + \beta X_2^2 + \gamma Y_1^2 + 2\delta X_1 X_2 + 2\epsilon X_1 Y_1 + 2\phi X_2 Y_1$$
$$+ 2\nu X_2 + 2\lambda Y_1 + 2\theta X_1 + \tau$$

where α, β, γ, δ, ϵ, ϕ, ν, λ, θ, and τ are constants.

If the following conditions

1. deg $L_V\psi = 2$
2. The terms of second degree of ψ compose a positive-definite quadratic form
3. The terms of second degree of $L_V\psi$ compose a negative-definite quadratic form

are imposed, then Ω_ψ is bounded by an ellipsoid.

The Lie derivative of ψ with respect to V yields

$$L_V\psi = (-\mu X_1 + Y_1 X_2)(2\alpha X_1 + 2\delta X_2 + 2\epsilon Y_1 + 2\theta)$$
$$+(-\mu X_2 + (Y_1 - A)X_1)(2\beta X_2 + 2\delta X_1 + 2\phi Y_1 + 2\nu)$$
$$+(1 - X_1 X_2)(2\gamma Y_1 + 2\epsilon X_1 + 2\phi X_2 + 2\lambda)$$

From $L_V\psi = 0$ and condition 1 we find

$$\gamma = \alpha + \beta \qquad \text{and} \qquad \delta = \epsilon = \phi = 0 \tag{3}$$

From condition 2 we find that

$$\alpha > 0 \qquad \text{and} \qquad \beta > 0 \tag{4}$$

From condition 3 it follows that the cubic equation

$$-w^3 - 2\mu(\alpha + \beta)w^2 + ((\lambda + A\beta)^2 + \theta^2 + \nu^2 - 4\alpha\beta\mu^2)w + 2(\theta^2\alpha\mu + \nu^2\beta\mu - (\lambda + A\beta)\theta\nu) = 0$$

must only have negative solutions, so that

$$\begin{gathered} \theta^2\alpha\mu + \nu^2\beta\mu - (\lambda + A\beta)\theta\nu < 0, \\ (\lambda + A\beta)^2 + \theta^2 + \nu^2 - 4\alpha\beta\mu^2 < 0 \end{gathered} \tag{5}$$

Thus the conditions (3)–(5) provide us with the ellipsoid bounding. Thus ψ takes the form

$$\psi(X_1, X_2, Y_1) = \alpha X_1^2 + \beta X_2^2 + (\alpha + \beta)Y_1^2 + 2\nu X_2 + 2\lambda Y_1 + 2\theta X_1 + \tau$$

Now we show that it is not possible to satisfy the conditions. Condition (5) can be rewritten as

$$\mu < \frac{(\lambda + A\beta)\theta\nu}{\theta^2\alpha + \nu^2\beta}$$

$$\mu^2 > \frac{(\lambda + A\beta)^2 + \theta^2 + \nu^2}{4\alpha\beta}$$

From (4) and (5) we have $(\lambda + A\beta)\theta\nu > 0$ and

$$\frac{(\lambda + A\beta)^2 + \theta^2 + \nu^2}{4\alpha\beta} < \frac{(\lambda + A\beta)^2\theta^2\nu^2}{(\theta^2\alpha + \nu^2\beta)^2}$$

which gives

$$\theta^2 + \nu^2 < \frac{-(\lambda + A\beta)^2(\theta^2\alpha - \nu^2\beta)^2}{(\theta^2\alpha + \nu^2\beta)^2}$$

which cannot be satisfied for real θ, ν, λ, β, α, and A.

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